THE HILBERT-KUNZ FUNCTION IN GRADED DIMENSION TWO

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ABSTRACT. Let R denote a two-dimensional normal standard-graded K-domain over the algebraic closure K of a finite field of characteristic p, and let $I \subset R$ denote a homogeneous R_+-primary ideal. We prove that the Hilbert-Kunz function of I has the form $\varphi(q) = e_{HK}(I)q^2 + \gamma(q)$ with rational Hilbert-Kunz multiplicity $e_{HK}(I)$ and an eventually periodic function $\gamma(q)$.

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INTRODUCTION

Suppose that $(R, \mathfrak{m})$ is a local Noetherian or a standard-graded ring of dimension d containing a field K of positive characteristic p. Let I denote an $\mathfrak{m}$-primary ideal and set $I^{[q]} = (f^q : f \in I)$, $q = p^e$. The function $e \mapsto \lambda(R/I^{[p^e]})$, where λ denotes the length, is called the Hilbert-Kunz function of the ideal I and was first considered by Kunz in [9]. Monsky showed in [12] that this function has the form (we write q for the argument, not e)

$$\varphi(q) = e_{HK}(I)q^d + O(q^{d-1}) \,,$$

where $e_{HK}(I)$ is a positive real number called the Hilbert-Kunz multiplicity of the ideal. It is conjectured that the Hilbert-Kunz multiplicity is always a rational number.

In [7], Huneke, McDermott and Monsky studied further the Hilbert-Kunz function showing that

$$\varphi(q) = e_{HK}(I)q^d + \beta q^{d-1} + O(q^{d-2})$$

holds with another real number β under the condition that R is normal and excellent with a perfect residue field ([7, Theorem 1.12]).

In this paper we want to investigate the Hilbert-Kunz-function and in particular the $O(q^{d-2})$-term in the case of a two-dimensional normal standard-graded K-domain over an algebraically closed field K of positive characteristic p. In this case we have recently shown that the Hilbert-Kunz multiplicity is a rational number (see [2, Theorem 3.6], the rationality for $I = \mathfrak{m}$ was obtained independently by Trivedi in [14]), and Monsky announced in [7, Remark 2.5] a proof that the second coefficient $\beta = 0$ vanishes. Our main result is the following theorem.

Theorem 1. Let K denote the algebraic closure of a finite field of characteristic p. Let R denote a normal two-dimensional standard-graded K-domain and let I denote a homogeneous R_+-primary ideal. Then the Hilbert-Kunz function of I has the form

$$\varphi(q) = e_{HK}(I)\, q^2 + \gamma(q)\,,$$

where the Hilbert-Kunz multiplicity $e_{HK}(I)$ is a rational number and where $\gamma(q)$ is an eventually periodic function.

This result has been expected by several people, but proves are known so far only in the case of the maximal ideal in a regular ring ([4]) or for certain cones over elliptic curves ([3], [13], [5]). In these known cases the condition that K is the algebraic closure of a finite field is not needed.

We give an overview for the argument for this result and of this paper as a whole. Write $I = (f_1, \ldots, f_n)$ with homogeneous ideal generators f_i of degree $d_i = \deg(f_i)$. We shall use the short exact sequence

$$0 \longrightarrow \operatorname{Syz}(f_1^q, \ldots, f_n^q)(m) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}(m - qd_i) \stackrel{f_1^q,\ldots,f_n^q}{\longrightarrow} \mathcal{O}(m) \longrightarrow 0$$

on the smooth projective curve $Y = \operatorname{Proj} R$ to compute

$$\lambda((R/I^{[q]})_m) = h^0(\mathcal{O}(m)) - \sum_{i=1}^n h^0(\mathcal{O}(m - qd_i)) + h^0(\operatorname{Syz}(f_1^q, \ldots, f_n^q)(m))\,.$$

This gives the Hilbert-Kunz function by summing over m, which is a finite sum. The syzygy bundle $\operatorname{Syz}(f_1^q, \ldots, f_n^q)(m)$ is a locally free sheaf on Y and we have to compute its global sections for varying q and for m running in certain ranges.

It is natural and helpful to consider more generally an arbitrary locally free sheaf $\mathcal{S}$ on a smooth projective curve Y over an algebraically closed field K of positive characteristic endowed with a fixed very ample invertible sheaf $\mathcal{O}_Y(1)$. Then we have to understand the global sections $H^0(Y, \mathcal{S}^q(m))$, where $\mathcal{S}^q$ denotes the pull-back under the $e-$th absolute Frobenius morphism, $q = p^e$. The appropriate object to study here is the expression

$$\sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil - 1} h^0(\mathcal{S}^q(m))$$

for rational numbers σ and ρ. The reason for this setting is that is allows us to do induction on the strong Harder-Narasimhan filtration of $\mathcal{S}$. This is the filtration $\mathcal{S}_1^q \subset \ldots \subset \mathcal{S}_t^q = \mathcal{S}^q$ such that the quotients $\mathcal{S}_k^q/\mathcal{S}_{k-1}^q$ are strongly semistable of decreasing slopes. Strongly semistable means that every Frobenius pull-back is again semistable. Such a filtration exists and is stable for $q \gg 0$ due to a Theorem of Langer. Using this we can reduce many questions to the case where $\mathcal{S}$ itself is strongly semistable.

It turns out that the above expression is related to what we call the Hilbert-Kunz slope of $\mathcal{S}$ (section 1). This is by definition $\mu_{HK}(\mathcal{S}) = \sum_{k=1}^{t} r_k \bar{\mu}_k^2$, where the rational numbers $\bar{\mu}_k = \mu(\mathcal{S}_k^q/\mathcal{S}_{k-1}^q)/q$ come from the strong Harder-Narasimhan filtration. We get after some preparatory work in section 2 the following formula (Theorem 3.2).

Theorem 2. Let $\mathcal{S}$ denote a locally free sheaf on a smooth projective curve Y over an algebraically closed field K of positive characteristic p. Let $\mathcal{S}_1^q \subset \ldots \subset \mathcal{S}_t^q = \mathcal{S}^q$ denote the strong Harder-Narasimhan filtration of $\mathcal{S}$. Let $r_k = \operatorname{rk}(\mathcal{S}_k^q/\mathcal{S}_{k-1}^q)$, $\bar{\mu}_k = \mu(\mathcal{S}_k^q/\mathcal{S}_{k-1}^q)/q$ and $\nu_k = -\bar{\mu}_k/\deg(Y)$, $k = 1, \ldots, t$. Write $\lceil q\nu_k \rceil = q\nu_k + \pi_k$ with the eventually periodic functions $\pi_k = \pi_k(q)$. Let $\sigma \leq \nu_1$ and $\rho \gg \nu_t$ denote rational numbers and set $\lceil q\rho \rceil = q\rho + \pi$. Then for $q = p^e \gg 0$ we have $\sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil - 1} h^0(\mathcal{S}^q(m)) =$

$$
\begin{aligned}
= \quad & \frac{q^2}{2\deg(Y)}\Big(\mu_{HK}(\mathcal{S}) + 2\rho \deg(\mathcal{S}) \deg(Y) + \rho^2 \operatorname{rk}(\mathcal{S}) \deg(Y)^2\Big) \\
& + q(\rho \operatorname{rk}(\mathcal{S}) + \frac{\deg(\mathcal{S})}{\deg(Y)})(1 - g - \frac{\deg(Y)}{2}) \\
& + q\pi(\deg(\mathcal{S}) + \rho \operatorname{rk}(\mathcal{S}) \deg(Y)) \\
& + \operatorname{rk}(\mathcal{S})\pi\Big((\pi - 1)\frac{\deg(Y)}{2} + 1 - g\Big) - \sum_{k=1}^{t} r_k \pi_k \Big((\pi_k - 1)\frac{\deg(Y)}{2} + 1 - g\Big) \\
& + \sum_{k=1}^{t} \Big(\sum_{m=\lceil q\nu_k \rceil}^{\lceil q\nu_k \rceil + \lceil \frac{\deg(\omega)}{\deg(Y)} \rceil} h^1((\mathcal{S}_k/\mathcal{S}_{k-1})^q(m)) \Big)
\end{aligned}
$$

We emphasize that the right hand side is a simplification of the left hand side. It is up to the last h^1-term a quadratic polynomial in q, where the linear term and the constant term have eventually periodic coefficients determined by the eventually periodic functions π and π_k. In general we only know that the h^1-term is a bounded function of q (Lemma 4.1). If however the groundfield K is the algebraic closure of a finite field, then this term is also eventually periodic (Theorem 4.3). This is due to the fact that the degree of the occurring strongly semistable quotient sheaves $(\mathcal{S}_k/\mathcal{S}_{k-1})^q(m)$ vary in a finite range, hence they form a bounded family. Since they are defined over a finite field they behave eventually periodically in q.

In section 5 we look at a short exact sequence $0 \to \mathcal{S} \to \mathcal{T} \to \mathcal{Q} \to 0$ and consider the alternating sum

$$\sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil - 1} h^0(\mathcal{S}^q(m)) - \sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil - 1} h^0(\mathcal{T}^q(m)) + \sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil - 1} h^0(\mathcal{Q}^q(m))$$

for suitable σ and ρ (or summing over $\mathbb{Z}$). We shall see that this equals

$$\frac{q^2}{2\deg(Y)}(\mu_{HK}(\mathcal{S}) - \mu_{HK}(\mathcal{T}) + \mu_{HK}(\mathcal{Q})) + O(q^0)$$

and that the $O(q^0)$-term is eventually periodic if everything is defined over a finite field (Theorem 5.1). This result underlines the significance of the Hilbert-Kunz slope and shows that the expression $(\mu_{HK}(\mathcal{S}) - \mu_{HK}(\mathcal{T}) + \mu_{HK}(\mathcal{Q}))/2\deg(Y)$ is an important invariant of a short exact sequences, which might be called its Hilbert-Kunz multiplicity.

In the last section 6 we apply this to the short exact syzygy sequence $0 \to \operatorname{Syz}(f_1, \ldots, f_n)(0) \to \bigoplus_{i=1}^n \mathcal{O}(-d_i) \to \mathcal{O}(0) \to 0$ given by ideal generators f_i and obtain Theorem 1 (Theorem 6.1). The Hilbert-Kunz multiplicity of this sequence equals the Hilbert-Kunz multiplicity of the ideal.

1. The Hilbert-Kunz slope in positive characteristic

Let Y denote a smooth projective curve over an algebraically closed field K. We recall briefly some notions about vector bundles, see [8] for details. The degree of a locally free sheaf $\mathcal{S}$ on Y of rank r is defined by $\deg(\mathcal{S}) = \deg \bigwedge^r(\mathcal{S})$. The slope of $\mathcal{S}$, written $\mu(\mathcal{S})$, is defined by $\deg(\mathcal{S})/r$. The slope has the property that $\mu(\mathcal{S} \otimes \mathcal{T}) = \mu(\mathcal{S}) + \mu(\mathcal{T})$.

A locally free sheaf $\mathcal{S}$ is called semistable if $\mu(\mathcal{T}) \leq \mu(\mathcal{S})$ holds for every locally free subsheaf $\mathcal{T} \subseteq \mathcal{S}$. Dualizing and tensoring with an invertible sheaf does not affect this property.

For every locally free sheaf $\mathcal{S}$ on Y there exists the so-called Harder-Narasimhan filtration $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t = \mathcal{S}$, where the $\mathcal{S}_k$ are locally free subsheaves. This filtration is unique and has the property that the quotients $\mathcal{S}_k/\mathcal{S}_{k-1}$ are semistable and $\mu(\mathcal{S}_k/\mathcal{S}_{k-1}) > \mu(\mathcal{S}_{k+1}/\mathcal{S}_k)$ holds.

The number $\mu(\mathcal{S}_1) = \mu_{\max}(\mathcal{S})$ is called the maximal slope of $\mathcal{S}$, and the minimal slope of $\mathcal{S}$ is $\mu_{\min}(\mathcal{S}) = \mu(\mathcal{S}/\mathcal{S}_{t-1})$. The existence of global sections can be tested with the maximal slope: if $\mu_{\max}(\mathcal{S}) < 0$, then $H^0(Y, \mathcal{S}) = 0$. Furthermore we have the relation $\mu_{\max}(\mathcal{S}) = -\mu_{\min}(\mathcal{S}^\vee)$, where $\mathcal{S}^\vee$ denotes the dual bundle.

With the help of the Harder-Narasimhan filtration we define in characteristic zero the Hilbert-Kunz slope of a locally free sheaf by

$$\mu_{HK}(\mathcal{S}) = \sum_{k=1}^{t} r_k \mu_k^2 \,,$$

where $r_k = \operatorname{rk}(\mathcal{S}_k/\mathcal{S}_{k-1})$ and $\mu_k = \mu(\mathcal{S}_k/\mathcal{S}_{k-1})$. See [1] for the basic properties of this notion and its relation to solid closure.

In positive characteristic we need the strong Harder-Narasimhan filtration. We denote the pull-back of $\mathcal{S}$ under the absolute Frobenius $F^e : Y \to Y$ by $\mathcal{S}^q$, $q = p^e$. A locally free sheaf $\mathcal{S}$ is called strongly semistable if $\mathcal{S}^q$ is semistable for every q. Due to a theorem of A. Langer [11, Theorem 2.7] there exists a Frobenius power such that the quotients in the Harder-Narasimhan filtration of the pull-back $\mathcal{S}^q$ are all strongly semistable. We call such a

filtration the strong Harder-Narasimhan filtration and denote it by

$$0 \subset \mathcal{S}_1^q \subset \ldots \subset \mathcal{S}_t^q = \mathcal{S}^q .$$

For $q' \geq q \gg 0$ the Harder-Narasimhan filtration of $\mathcal{S}^{q'}$ is

$$\mathcal{S}_1^{q'} = (\mathcal{S}_1^q)^{q'/q} \subset \ldots \subset (\mathcal{S}_t^{q'})^{q'/q} = (\mathcal{S}_t^q)^{q'/q} .$$

This allows to define rational numbers $\bar{\mu}_k = \bar{\mu}_k(\mathcal{S}) = \frac{\mu(\mathcal{S}_k^q/\mathcal{S}_{k-1}^q)}{q}$ for $q \gg 0$. The length t of the strong Harder-Narasimhan filtration as well as the ranks $r_k = \operatorname{rk}(\mathcal{S}_k^q/\mathcal{S}_{k-1}^q)$ are independent of $q \gg 0$. For $q \gg 0$ we have $\mu(\mathcal{S}_k^q/\mathcal{S}_{k-1}^q) = q\bar{\mu}_k$.

We now define the Hilbert-Kunz slope in positive characteristic.

Definition 1.1. Let $\mathcal{S}$ denote a locally free sheaf on a smooth projective curve over an algebraically closed field of positive characteristic. Let $\mathcal{S}_1^q \subset \ldots \subset \mathcal{S}_t^q = \mathcal{S}^q$ denote the strong Harder-Narasimhan filtration of $\mathcal{S}$. Then the Hilbert-Kunz slope of $\mathcal{S}$ is

$$\mu_{HK}(\mathcal{S}) = \sum_{k=1}^{t} r_k \bar{\mu}_k^2 .$$

The Hilbert-Kunz slope is a rational number. With this notion we may express the formula for the Hilbert-Kunz multiplicity in the following way ([2, Theorem 3.6]).

Theorem 1.2. *Let R denote a two-dimensional standard-graded normal domain and let $I = (f_1, \ldots, f_n)$ denote a homogeneous R_+-primary ideal generated by homogeneous elements f_i of degree $d_i, i = 1, \ldots, n$. Then the Hilbert-Kunz multiplicity $e_{HK}(I)$ equals*

$$e_{HK}(I) = \frac{1}{2\deg(Y)}(\mu_{HK}\Big(\operatorname{Syz}(f_1, \ldots, f_n)(0)) - \deg(Y)^2 \sum_{i=1}^{n} d_i^2\Big) .$$

We shall see in the next section that the Hilbert-Kunz slope controls in general the quadratic behavior (the q^2 term) of the sections $\sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil -1} h^0(\mathcal{S}^q(m))$, where ρ and σ are rational numbers.

2. Dimension of sections

Lets fix the notation for this and the following sections. Let K denote an algebraically closed field of positive characteristic p. Let Y denote a smooth projective curve over K of genus g and canonical sheaf ω. We fix a very ample invertible sheaf $\mathcal{O}(1)$ and denote by $\deg(Y) = \deg(\mathcal{O}(1))$ the degree of the curve. As usual we set $\mathcal{S}(m) = \mathcal{S} \otimes \mathcal{O}(m)$ for a coherent sheaf $\mathcal{S}$ on Y. We shall only consider locally free sheaves.

We want to describe the sum $\sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil -1} h^0(\mathcal{S}^q(m))$ for a locally free sheaf $\mathcal{S}$, where σ and ρ are rational numbers. It will become clear during this paper

that we cannot avoid rational boundaries. In fact we should be lucky that only rational boundaries occur. We may write $\lceil q\rho \rceil = q\rho + \pi(q)$, where $0 \leq \pi(q) < 1$. We often shall write π instead of $\pi(q)$. If $\rho = a/b$, then $\lceil \frac{p^e a}{b} \rceil = \frac{p^e a}{b} + \frac{(-p^e a) \bmod b}{b}$, $0 \leq (-p^e a) \bmod b < b$, hence $\pi(q) = \frac{(-p^e a) \bmod b}{b}$ is an eventually periodic function. This is one source (the rounding source) of the periodic behavior of the Hilbert-Kunz function, we treat the other (the H^1 source) in section 4.

Lemma 2.1. *Let $\mathcal{S}$ denote a locally free sheaf on Y. Let $\sigma < \rho$ denote rational numbers. Write $\lceil q\sigma \rceil = q\sigma + \delta$ and $\lceil q\rho \rceil = q\rho + \pi$. Then*

$$\begin{aligned}
\sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil -1} h^0(\mathcal{S}^q(m)) &= q^2\Big((\rho-\sigma)\deg(\mathcal{S}) + (\rho^2-\sigma^2)\frac{\operatorname{rk}(\mathcal{S})\deg(Y)}{2}\Big) \\
&\quad + q(\rho-\sigma)\operatorname{rk}(\mathcal{S})(1-g-\frac{\deg(Y)}{2}) \\
&\quad + q\Big((\pi-\delta)\deg(\mathcal{S}) + (\pi\rho-\delta\sigma)\operatorname{rk}(\mathcal{S})\deg(Y)\Big) \\
&\quad + \operatorname{rk}(\mathcal{S})\Big((\pi(\pi-1)-\delta(\delta-1))\frac{\deg(Y)}{2} + (\pi-\delta)(1-g)\Big) \\
&\quad + \sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil -1} h^1(\mathcal{S}^q(m))
\end{aligned}$$

Proof. Due to the formula of Riemann-Roch we have

$$\begin{aligned}
\sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil -1} h^0(\mathcal{S}^q(m)) &= \sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil -1} \deg(\mathcal{S}^q(m)) + \sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil -1} \operatorname{rk}(\mathcal{S})(1-g) + \sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil -1} h^1(\mathcal{S}^q(m)) \\
&= \sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil -1} \Big(q\deg(\mathcal{S}) + m\operatorname{rk}(\mathcal{S})\deg(Y)\Big) \\
&\quad + (\lceil q\rho \rceil - \lceil q\sigma \rceil)\operatorname{rk}(\mathcal{S})(1-g) + \sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil -1} h^1(\mathcal{S}^q(m)) \\
&= q(\lceil q\rho \rceil - \lceil q\sigma \rceil)\deg(\mathcal{S}) \\
&\quad + \Big(\lceil q\rho \rceil(\lceil q\rho \rceil - 1) - \lceil q\sigma \rceil(\lceil q\sigma \rceil - 1)\Big)\frac{\operatorname{rk}(\mathcal{S})\deg(Y)}{2} \\
&\quad + (\lceil q\rho \rceil - \lceil q\sigma \rceil)\operatorname{rk}(\mathcal{S})(1-g) + \sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil -1} h^1(\mathcal{S}^q(m))\,.
\end{aligned}$$

We now insert $\lceil q\sigma \rceil = q\sigma + \delta(q)$ and $\lceil q\rho \rceil = q\rho + \pi(q)$ in these summands. The first summand yields

$$q(q(\rho-\sigma)+\pi-\delta)\deg(\mathcal{S}) = q^2(\rho-\sigma)\deg(\mathcal{S}) + q(\pi-\delta)\deg(\mathcal{S})\,.$$

The second summand yields

$$\Big(q^2(\rho^2-\sigma^2) - q(\rho-\sigma) + q(2\pi\rho - 2\delta\sigma) + \pi(\pi-1) - \delta(\delta-1)\Big)\frac{\operatorname{rk}(\mathcal{S})\deg(Y)}{2}\,.$$

The third summand yields

$$(q(\rho-\sigma)+\pi-\delta)\operatorname{rk}(\mathcal{S})(1-g)\,.$$

We regroup the terms and get the q^2-term

$$q^2(\rho-\sigma)\deg(\mathcal{S})+q^2(\rho^2-\sigma^2)\frac{\operatorname{rk}(\mathcal{S})\deg(Y)}{2}\,,$$

the constant q-term

$$-q(\rho-\sigma)\frac{\deg(Y)\operatorname{rk}(\mathcal{S})}{2}+q(\rho-\sigma)\operatorname{rk}(\mathcal{S})(1-g)=q(\rho-\sigma)\operatorname{rk}(\mathcal{S})(1-g-\frac{\deg(Y)}{2})\,,$$

the periodic q-term

$$q(\pi-\delta)\deg(\mathcal{S})+q(2\pi\rho-2\delta\sigma)\frac{\operatorname{rk}(\mathcal{S})\deg(Y)}{2}$$

and the periodic q^0-term

$$(\pi(\pi-1)-\delta(\delta-1))\frac{\operatorname{rk}(\mathcal{S})\deg(Y)}{2}+(\pi-\delta)\operatorname{rk}(\mathcal{S})(1-g)\,.$$

This is what we have written down. □

This Lemma is of course only useful if we can say something about the h^1-term. We treat first the case of a strongly semistable sheaf $\mathcal{S}$.

Proposition 2.2. *Let $\mathcal{S}$ denote a strongly semistable sheaf on Y. Set $\nu=-\mu(\mathcal{S})/\deg(Y)=-\deg(\mathcal{S})/\operatorname{rk}(\mathcal{S})\deg(Y)$. Let σ and ρ denote rational numbers such that $\sigma\leq\nu\ll\rho$. Let $\lceil q\nu\rceil=q\nu+\epsilon$ and $\lceil q\rho\rceil=q\rho+\pi$. Then*

$$\begin{aligned}\sum_{m=\lceil q\sigma\rceil}^{\lceil q\rho\rceil-1}h^0(\mathcal{S}^q(m))=&\frac{q^2}{2\deg(Y)}\Big(\mu_{HK}(\mathcal{S})+2\rho\deg(\mathcal{S})\deg(Y)+\rho^2\operatorname{rk}(\mathcal{S})\deg(Y)^2\Big)\\&+q(\rho\operatorname{rk}(\mathcal{S})+\frac{\deg(\mathcal{S})}{\deg(Y)})(1-g-\frac{\deg(Y)}{2})\\&+q\pi\Big(\deg(\mathcal{S})+\rho\operatorname{rk}(\mathcal{S})\deg(Y)\Big)\\&+\operatorname{rk}(\mathcal{S})\Big((\pi(\pi-1)-\epsilon(\epsilon-1))\frac{\deg(Y)}{2}+(\pi-\epsilon)(1-g)\Big)\\&+\sum_{m=\lceil q\nu\rceil}^{\lceil q\nu\rceil+\lceil\frac{\deg(\omega)}{\deg(Y)}\rceil}h^1(\mathcal{S}^q(m))\end{aligned}$$

Proof. For $m<\lceil q\nu\rceil=\lceil-q\mu(\mathcal{S})/\deg(Y)\rceil$ we have $m<-q\mu(\mathcal{S})/\deg(Y)$ and therefore

$$\deg(\mathcal{S}^q(m))=q\deg(\mathcal{S})+m\operatorname{rk}(\mathcal{S})\deg(Y)<0\,.$$

Since $\mathcal{S}^q$ is semistable we have $h^0(\mathcal{S}^q(m))=0$ in this range. Therefore we have

$$\sum_{m=\lceil q\sigma\rceil}^{\lceil q\rho\rceil-1}h^0(\mathcal{S}^q(m))=\sum_{m=\lceil q\nu\rceil}^{\lceil q\rho\rceil-1}h^0(\mathcal{S}^q(m))\,.$$

We apply the formula from Lemma 2.1 to the right hand side. We insert $\nu = -\frac{\deg(\mathcal{S})}{\operatorname{rk}(\mathcal{S})\deg(Y)}$ and get the quadratic term

$$\begin{aligned}
& \deg(\mathcal{S})\big(\rho + \frac{\deg(\mathcal{S})}{\operatorname{rk}(\mathcal{S})\deg(Y)}\big) + \frac{\operatorname{rk}(\mathcal{S})\deg(Y)}{2}\big(\rho^2 - \frac{\deg(\mathcal{S})^2}{\operatorname{rk}(\mathcal{S})^2\deg(Y)^2}\big) \\
= \;& \rho\deg(\mathcal{S}) + \rho^2\frac{\operatorname{rk}(\mathcal{S})\deg(Y)}{2} + \frac{\deg(\mathcal{S})^2}{\operatorname{rk}(\mathcal{S})\deg(Y)} - \frac{1}{2}\frac{\deg(\mathcal{S})^2}{\operatorname{rk}(\mathcal{S})\deg(Y)} \\
= \;& \frac{1}{2\deg(Y)}\big(2\rho\deg(\mathcal{S})\deg(Y) + \rho^2\operatorname{rk}(\mathcal{S})\deg(Y)^2 + \frac{\deg(\mathcal{S})^2}{\operatorname{rk}(\mathcal{S})}\big)\,.
\end{aligned}$$

The linear-constant term yields

$$\big(\rho + \frac{\deg(\mathcal{S})}{\operatorname{rk}(\mathcal{S})\deg(Y)}\big)\operatorname{rk}(\mathcal{S})(1-g-\frac{\deg(Y)}{2})\,.$$

The linear-periodic term is

$$\begin{aligned}
= \;& (\pi-\epsilon)\deg(\mathcal{S}) + (\pi\nu + \frac{\epsilon\deg(\mathcal{S})}{\operatorname{rk}(\mathcal{S})\deg(Y)})\operatorname{rk}(\mathcal{S})\deg(Y) \\
= \;& (\pi-\epsilon)\deg(\mathcal{S}) + \pi\nu\operatorname{rk}(\mathcal{S})\deg(Y) + \epsilon\deg(\mathcal{S}) \\
= \;& \pi\Big(\deg(\mathcal{S}) + \nu\operatorname{rk}(\mathcal{S})\deg(Y)\Big)\,.
\end{aligned}$$

The forth term comes directly from the formula in Lemma 2.1. So now we have to look at $\sum_{m=\lceil q\nu\rceil}^{\lceil q\rho\rceil-1} h^1(\mathcal{S}^q(m))$. For $m > \lceil q\nu\rceil + \lceil\frac{\deg(\omega)}{\deg(Y)}\rceil$ we have

$$\mu((\mathcal{S}^q)^\vee(-m)\otimes\omega) = -q\mu(\mathcal{S}) - m\deg(Y) + \deg(\omega) < 0\,,$$

hence $h^1(\mathcal{S}^q(m)) = 0$ by Serre duality and the semistability of $\mathcal{S}^q$. $\square$

3. Dimension of sections - general case

We shall extend the results of the previous section to arbitrary locally free sheaves using the strong Harder-Narasimhan filtration. If $\mathcal{S}_1^q \subset \ldots \subset \mathcal{S}_t^q = \mathcal{S}^q$ is the strong Harder-Narasimhan filtration of $\mathcal{S}$, we set $\bar{\mu}_k = \mu(\mathcal{S}_k^q/\mathcal{S}_{k-1}^q)/q$ and $\nu_k = -\bar{\mu}_k/\deg(Y)$.

Lemma 3.1. *Let $\mathcal{S}$ denote a locally free sheaf on a smooth projective curve Y over an algebraically closed field K of positive characteristic p. Let $\mathcal{S}_k^q \subset \mathcal{S}^q$ denote the strong Harder-Narasimhan filtration of $\mathcal{S}$. Then for numbers $\sigma \leq \nu_1$ and $\rho > \nu_t$ and for $q = p^e \gg 0$ we have*

$$\sum_{m=\lceil q\sigma\rceil}^{\lceil q\rho\rceil-1} h^0(\mathcal{S}^q(m)) = \sum_{k=1}^{t}\Big(\sum_{m=\lceil q\nu_k\rceil}^{\lceil q\rho\rceil-1} h^0((\mathcal{S}_k/\mathcal{S}_{k-1})^q(m))\Big)\,.$$

Proof. We do induction on the strong Harder-Narasimhan filtration, so assume that $q \gg 0$ such that the Harder-Narasimhan filtration of $\mathcal{S}^q$ is strong. For $t = 1$, that is in the strongly semistable case, the statement was proved in the beginning of the proof of Proposition 2.2. We use the short exact

sequence $0 \to \mathcal{S}_{t-1} \to \mathcal{S} \to \mathcal{S}/\mathcal{S}_{t-1} \to 0$ and we assume that the statement is true for $\mathcal{S}_{t-1}$. We write

$$\sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil -1} h^0(\mathcal{S}^q(m)) = \sum_{m=\lceil q\sigma \rceil}^{\lceil q\nu_t \rceil -1} h^0(\mathcal{S}^q(m)) + \sum_{m=\lceil q\nu_t \rceil}^{\lceil q\rho \rceil -1} h^0(\mathcal{S}^q(m)) \,.$$

Look at the two summands on the right. For the first summand we have $m \leq \lceil q\nu_t \rceil - 1$, hence $m < q\nu_t = -q\bar{\mu}_t / \deg(Y)$ and therefore

$$\mu((\mathcal{S}/\mathcal{S}_{t-1})^q(m)) = q\bar{\mu}_t + m\deg(Y) < 0 \,.$$

Hence $h^0((\mathcal{S}/\mathcal{S}_{t-1})^q(m)) = 0$ and therefore the short exact sequence yields $\sum_{m=\lceil q\sigma \rceil}^{\lceil q\nu_t \rceil -1} h^0(\mathcal{S}^q(m)) = \sum_{m=\lceil q\sigma \rceil}^{\lceil q\nu_t \rceil -1} h^0(\mathcal{S}_{t-1}^q(m))$.

Now look at the second summand. Since $m \geq q\nu_t$ we have $m > q\nu_{t-1} + \frac{\deg(\omega)}{\deg(Y)}$ for $q \gg 0$. Therefore we have

$$\mu_{\max}((\mathcal{S}_{t-1}^q)^\vee(-m) \otimes \omega) = -q\bar{\mu}_{t-1} - m\deg(Y) + \deg(\omega) < 0$$

and so $H^1(\mathcal{S}_{t-1}^q(m)) = 0$ in this range. Thus

$$\sum_{m=\lceil q\nu_t \rceil}^{\lceil q\rho \rceil -1} h^0(\mathcal{S}^q(m)) = \sum_{m=\lceil q\nu_t \rceil}^{\lceil q\rho \rceil -1} h^0(\mathcal{S}_{t-1}^q(m)) + \sum_{m=\lceil q\nu_t \rceil}^{\lceil q\rho \rceil -1} h^0((\mathcal{S}/\mathcal{S}_{t-1})^q(m)) \,.$$

This gives the result. $\square$

The following theorem describes the global sections of a locally free sheaf.

Theorem 3.2. *Let $\mathcal{S}$ denote a locally free sheaf on a smooth projective curve Y over an algebraically closed field K of positive characteristic p. Let $\mathcal{S}_1^q \subset \ldots \subset \mathcal{S}_t^q = \mathcal{S}^q$ denote the strong Harder-Narasimhan filtration of $\mathcal{S}$. Let $r_k = \operatorname{rk}(\mathcal{S}_k^q/\mathcal{S}_{k-1}^q)$, $\bar{\mu}_k = \mu(\mathcal{S}_k^q/\mathcal{S}_{k-1}^q)/q$ and $\nu_k = -\bar{\mu}_k/\deg(Y)$. Write $\lceil q\nu_k \rceil = q\nu_k + \pi_k$ with the eventually periodic functions $\pi_k = \pi_k(q)$. Let $\sigma \leq \nu_1$ and $\rho \gg \nu_t$ denote rational numbers and set $\lceil q\rho \rceil = q\rho + \pi$. Then for $q = p^e \gg 0$ we have $\sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil -1} h^0(\mathcal{S}^q(m)) =$*

$$\begin{aligned}
= \;\; & \frac{q^2}{2\deg(Y)}\Big(\mu_{HK}(\mathcal{S}) + 2\rho\deg(\mathcal{S})\deg(Y) + \rho^2\operatorname{rk}(\mathcal{S})\deg(Y)^2\Big) \\
& + q(\rho\operatorname{rk}(\mathcal{S}) + \frac{\deg(\mathcal{S})}{\deg(Y)})(1 - g - \frac{\deg(Y)}{2}) \\
& + q\pi(\deg(\mathcal{S}) + \rho\operatorname{rk}(\mathcal{S})\deg(Y)) \\
& + \operatorname{rk}(\mathcal{S})\pi\Big((\pi - 1)\frac{\deg(Y)}{2} + 1 - g\Big) - \sum_{k=1}^{t} r_k\pi_k\Big((\pi_k - 1)\frac{\deg(Y)}{2} + 1 - g\Big) \\
& + \sum_{k=1}^{t}\Big(\sum_{m=\lceil q\nu_k \rceil}^{\lceil q\nu_k \rceil + \lceil \frac{\deg(\omega)}{\deg(Y)} \rceil} h^1((\mathcal{S}_k/\mathcal{S}_{k-1})^q(m))\Big)
\end{aligned}$$

Proof. Let q be big enough such that the statement of Lemma 3.1 holds true, so that we have

$$\sum_{m=\lceil q\sigma\rceil}^{\lceil q\rho\rceil-1} h^0(\mathcal{S}^q(m)) = \sum_{k=1}^{t}\Big(\sum_{m=\lceil q\nu_k\rceil}^{\lceil q\rho\rceil-1} h^0((\mathcal{S}_k/\mathcal{S}_{k-1})^q(m))\Big)\,.$$

For $t=1$, that is in the strongly semistable case, the result is just Proposition 2.2 (with $\epsilon=\pi_1$). In general we have to sum over the t expressions coming from Proposition 2.2 for the strongly semistable quotient sheaves $\mathcal{S}_1^q, \mathcal{S}_2^q/\mathcal{S}_1^q, \ldots, \mathcal{S}^q/\mathcal{S}_{t-1}^q$. The rank and the degree are additive on short exact sequences and the Hilbert-Kunz slope is additive on the quotients in the strong Harder-Narasimhan filtration. Therefore the summations of the first, the second and the third terms from Proposition 2.2 yield the first, the second and the third expression in the statement. This is also true for the fifth term. For the forth term we just have to add

$$\begin{aligned}&\sum_{k=1}^{t} r_k\Big((\pi(\pi-1)-\pi_k(\pi_k-1))\frac{\deg(Y)}{2}+(\pi-\pi_k)(1-g)\Big)\\ =&\,\mathrm{rk}(\mathcal{S})\Big(\pi(\pi-1)\frac{\deg(Y)}{2}+\pi(1-g)\Big)-\sum_{k=1}^{t} r_k\Big(\pi_k(\pi_k-1)\frac{\deg(Y)}{2}+\pi_k(1-g)\Big)\end{aligned}$$

□

4. Boundedness and periodicity

We take now a closer look at the lower terms in the formula of Theorem 3.2.

Lemma 4.1. *Let $\mathcal{S}$ denote a strongly semistable locally free sheaf on the smooth projective curve Y over the algebraically closed field K of positive characteristic p. Set $\nu=-\frac{\deg(\mathcal{S})}{\mathrm{rk}(\mathcal{S})\deg(Y)}$ and let $\rho\geq\nu$ be a rational number. Then $\sum_{m=\lceil q\nu\rceil}^{\lceil q\rho\rceil} h^1(\mathcal{S}^q(m))=O(q^0)$.*

Proof. It is enough to consider the sum running from $\lceil q\nu\rceil$ to $\lceil q\nu\rceil+\lceil\frac{\deg(\omega)}{\deg(Y)}\rceil$, since above this we have $H^1(Y,\mathcal{S}^q(m))=0$. Hence we are concerned with $\lceil\frac{\deg(\omega)}{\deg(Y)}\rceil+1$ summands. From $\frac{-q\deg(\omega)}{\mathrm{rk}(\mathcal{S})\deg(Y)}\leq\lceil q\nu\rceil\leq m\leq\lceil q\nu\rceil+\lceil\frac{\deg(\omega)}{\deg(Y)}\rceil\leq q\nu+\frac{\deg(\omega)}{\deg(Y)}+2$ we get $0\leq\deg(\mathcal{S}^q(m))\leq\mathrm{rk}(\mathcal{S})(\deg(\omega)+2\deg(Y))$. Hence the degrees of these sheaves vary in a finite range.

It follows now from fundamental boundedness results for semistable sheaves that there exists an upper bound for the dimension of global sections of these sheaves ([8, Corollary 1.7.7]). This is then also true for $h^1(\mathcal{S}^q(m))$. □

Remark 4.2. In the construction via quot-schemes of the moduli space of semistable bundles of given degree one shows that there exists a coherent sheaf $\mathcal{F}$ such that every $\mathcal{S}$ admits a surjection $\mathcal{F}\to\mathcal{S}\to 0$. From this it follows again that $h^1(\mathcal{S})$ is bounded. For invertible sheaves of fixed degree one may use the theorem of Clifford, see [6, Theorem IV.5.4], to show that there

exists a common bound for the dimension of their global sections. Without the condition semistable this conclusion does not hold, as the example $\mathcal{L} \oplus \mathcal{L}^{-1}$ shows.

Theorem 4.3. *Let K denote an algebraically closed field of positive characteristic p, let Y denote a smooth projective curve over K. Let $\mathcal{S}$ denote a locally free sheaf on Y. Let $\sigma \leq \nu_1$ and $\rho \gg \nu_t$ denote rational numbers. Then we have*

$$\sum_{m=\lceil q\sigma \rceil}^{\lceil q\rho \rceil - 1} h^0(\mathcal{S}^q(m)) = \alpha q^2 + \beta(q) q + \gamma(q)\,,$$

where $\alpha = \frac{\mu_{HK}(\mathcal{S}) + 2\rho \deg(\mathcal{S}) + \rho^2 \operatorname{rk}(\mathcal{S}) \deg(Y)^2}{2\deg(Y)}$ is a rational number, $\beta(q)$ is an eventually periodic function and $\gamma(q)$ is a bounded function (both with rational values).

If moreover K is the algebraic closure of a finite field, then $\gamma(q)$ is also an eventually periodic function.

Proof. The first statement follows directly from Theorem 3.2 and Lemma 4.1. For the second statement we only have to show that the expression

$$\sum_{k=1}^{t} \Big(\sum_{m=\lceil q\nu_k \rceil}^{\lceil q\nu_k \rceil + \lceil \frac{\deg(\omega)}{\deg(Y)} \rceil} h^1(Y, (\mathcal{S}_k/\mathcal{S}_{k-1})^q(m)) \Big)$$

is an eventually periodic function in q. Thus we may assume that $\mathcal{S}$ is strongly semistable.

Set $\nu = -\frac{\deg(\mathcal{S})}{\operatorname{rk}(\mathcal{S}) \deg(Y)}$. We consider the starting term of the summation, $m(q) = \lceil q\nu \rceil$. Write $m(q) = q\nu + \pi(q)$ with the eventually periodic function $\pi(q)$. Let $\tilde{q}$ be the length of the periodicity. We have

$$\deg(\mathcal{S}^q(m(q))) = q\deg(\mathcal{S}) + m(q) \operatorname{rk}(\mathcal{S}) \deg(Y) = \pi(q) \operatorname{rk}(\mathcal{S}) \deg(Y)\,,$$

so that the degree of these sheaves behaves also eventually periodical with the same periodicity $\tilde{q}$.

We consider now a subset of type $M = \{q_0 \tilde{q}^\ell : \ell \in \mathbb{N}\}$. In particular $\pi(q)$ is constant on this set M. For $q \in M$ we have

$$\begin{aligned}
\mathcal{S}^{q\tilde{q}}(m(q\tilde{q})) &= \mathcal{S}^{q\tilde{q}} \otimes \mathcal{O}(q\tilde{q}\nu + \pi(q\tilde{q})) \\
&= (\mathcal{S}^q)^{\tilde{q}} \otimes \mathcal{O}(q\tilde{q}\nu + \pi(q) + (\tilde{q}-1)\pi(q)) \otimes \mathcal{O}((-\tilde{q}+1)\pi(q)) \\
&= (\mathcal{S}^q)^{\tilde{q}}(q\tilde{q}\nu + \tilde{q}\pi(q)) \otimes \mathcal{O}((-\tilde{q}+1)\pi(q)) \\
&= (\mathcal{S}^q(q\nu + \pi(q)))^{\tilde{q}} \otimes \mathcal{O}((-\tilde{q}+1)\pi(q))
\end{aligned}$$

This means that the $\tilde{q}$ successor is build from its predecessor by pulling it back and tensor the result with a fixed invertible sheaf. In particular this recursion rule is independent of q. Now the curve and the locally free sheaf $\mathcal{S}$ are defined over a finite subfield $L \subset K$. Then all the $\mathcal{S}^q(m(q))$, $q \in M$, have the same degree and are defined over L. The family of semistable sheaves

with fixed degree is bounded and therefore there exist only finitely many such sheaves defined over L. Hence the recursion above shows that the sequence of sheaves $\mathcal{S}^q(m(q))$, $q \in M$, is eventually periodic.

From this it follows immediately that also the other sheaves $\mathcal{S}^q(m(q)+s) = \mathcal{S}^q(m(q)) \otimes \mathcal{O}(s)$ for fixed s, $0 \leq s \leq \lceil \frac{\deg(\omega)}{\deg(Y)} \rceil$, occur periodically in $q \in N$. This is then also true for their h^1-term. □

Remark 4.4. A similar argument was used by Lange and Stuhler in [10] to show that the Frobenius pull-backs of a strongly semistable sheaf of degree 0 on a curve over a finite field behave eventually periodically.

5. Short exact sequences

In this section we look at a short exact sequence $0 \to \mathcal{S} \to \mathcal{T} \to \mathcal{Q} \to 0$ of locally free sheaves on a smooth projective curve Y over an algebraically closed field K of positive characteristic p. We want to compute the alternating sum

$$\sum_{m \in \mathbb{Z}} \Big(h^0(\mathcal{S}^q(m)) - h^0(\mathcal{T}^q(m)) + h^0(\mathcal{Q}^q(m)) \Big)$$

in dependence of $q = p^e$. We will see in the next section that the computation of the Hilbert-Kunz function of an ideal in a two-dimensional normal standard-graded domain is a special case of this consideration. The sum is for every q finite, since for $m \ll 0$ all terms are 0 and for $m \gg 0$ we have $H^1(Y, \mathcal{S}^q(m)) = 0$ and the sum is 0. The sum is the dimension of the cokernel $\sum_{m \in \mathbb{Z}} \dim(\Gamma(Y, \mathcal{Q}^q(m)) / \operatorname{im}(\Gamma(Y, \mathcal{T}^q(m)))$ and equals also

$$\sum_{m \in \mathbb{Z}} \Big(h^1(\mathcal{S}^q(m)) - h^1(\mathcal{T}^q(m)) + h^1(\mathcal{Q}^q(m)) \Big) .$$

The following theorem shows that this alternating sum is a quadratic polynomial with the alternating sum of the Hilbert-Kunz slopes as leading coefficient, with vanishing linear term and with a bounded constant term, which is periodic in the finite case.

Theorem 5.1. *Let Y denote a smooth projective curve over an algebraically closed field of positive characteristic p. Let $0 \to \mathcal{S} \to \mathcal{T} \to \mathcal{Q} \to 0$ denote a short exact sequence of locally free sheaves on Y. Then the following hold.*

(i) *The alternating sum of the global sections is*
$$\begin{aligned} &\sum_{m \in \mathbb{Z}} \Big(h^0(\mathcal{S}^q(m)) - h^0(\mathcal{T}^q(m)) + h^0(\mathcal{Q}^q(m)) \Big) \\ = \; &\frac{q^2}{2\deg(Y)} (\mu_{HK}(\mathcal{S}) - \mu_{HK}(\mathcal{T}) + \mu_{HK}(\mathcal{Q})) + O(q^0) . \end{aligned}$$

(ii) *Let $\mathcal{S}_k \subseteq \mathcal{S}$, $\mathcal{T}_i \subseteq \mathcal{T}$ and $\mathcal{Q}_j \subseteq \mathcal{Q}$ denote the strong Harder-Narasimhan filtrations of these sheaves with disjoint index sets. Let r_k, $\bar{\mu}_k$, ν_k and π_k (r_i, r_j, ν_i, ν_j etc. respectively) denote the ranks, slopes and the*

eventually periodic functions corresponding to the strongly semistable quotients. Then the $O(q^0)$-term equals

$$-\sum_k r_k \pi_k \Big((\pi_k - 1) \frac{\deg(Y)}{2} + 1 - g \Big) + \sum_i r_i \pi_i \Big((\pi_i - 1) \frac{\deg(Y)}{2} + 1 - g \Big)$$
$$-\sum_j r_j \pi_j \Big((\pi_j - 1) \frac{\deg(Y)}{2} + 1 - g \Big) + \sum_k \Big(\sum_{m=\lceil q\nu_k \rceil}^{\lceil q\nu_k \rceil + \lceil \frac{\deg(\omega)}{\deg(Y)} \rceil} h^1((\mathcal{S}_k^q / \mathcal{S}_{k-1}^q)(m)) \Big)$$
$$-\sum_i \Big(\sum_{m=\lceil q\nu_i \rceil}^{\lceil q\nu_i \rceil + \lceil \frac{\deg(\omega)}{\deg(Y)} \rceil} h^1((\mathcal{T}_i^q / \mathcal{T}_{i-1}^q)(m)) \Big) + \sum_j \Big(\sum_{m=\lceil q\nu_j \rceil}^{\lceil q\nu_j \rceil + \lceil \frac{\deg(\omega)}{\deg(Y)} \rceil} h^1((\mathcal{Q}_j^q / \mathcal{Q}_{j-1}^q)(m)) \Big).$$

(iii) *If K is the algebraic closure of a finite field, then the $O(q^0)$-term is eventually periodic.*

Proof. Let σ and ρ be rational numbers such that $\sigma \leq \nu_i, \nu_j, \nu_k \ll \rho$ for all k, i, j. Then we may look at the finite alternating sum running from $\lceil q\sigma \rceil$ to $\lceil q\rho \rceil$. We only have to add the expressions in Theorem 3.2 for $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{Q}$. Since the rank and the degree are additive on short exact sequences most terms vanish. The remaining terms are $\frac{q^2}{2\deg(Y)}(\mu_{HK}(\mathcal{S}) - \mu_{HK}(\mathcal{T}) + \mu_{HK}(\mathcal{Q}))$ and the terms written down in (ii). The boundedness stated in (ii) follows from Lemma 4.1 and the periodicity statement in (iii) follows from Theorem 4.3. $\square$

6. The Hilbert-Kunz function of an ideal

We come back to the Hilbert-Kunz function of an ideal. Let R denote a normal standard-graded domain over an algebraically closed field of positive characteristic p and let $f_1, \ldots, f_n$ denote homogeneous generators of an R_+-primary ideal of degrees $d_1, \ldots, d_n$. These data give rise to the short exact sequence of locally free sheaves on $Y = \operatorname{Proj} R$,

$$0 \longrightarrow \operatorname{Syz}(f_1, \ldots, f_n)(m) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}(m - d_i) \xrightarrow{f_1, \ldots, f_n} \mathcal{O}(m) \longrightarrow 0\,.$$

The pull-back of this short exact sequence under the e-th absolute Frobenius morphism $F^e : Y \to Y$ yields

$$0 \longrightarrow (F^e(\operatorname{Syz}(f_1, \ldots, f_n)))(m) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}(m - qd_i) \xrightarrow{f_1^q, \ldots, f_n^q} \mathcal{O}(m) \longrightarrow 0\,.$$

Since R is normal the global sections $\Gamma(Y, -)$ of this sequence yield

$$0 \longrightarrow \Gamma(Y, \operatorname{Syz}(f_1^q, \ldots, f_n^q)(m)) \longrightarrow \bigoplus_{i=1}^n R_{m-qd_i} \xrightarrow{f_1^q, \ldots, f_n^q} R_m \longrightarrow \ldots$$

and the cokernel of the last mapping is just $(R/I^{[q]})_m$. Hence we have

$$\lambda((R/I^{[q]})_m) = h^0(\mathcal{O}(m)) - \sum_{i=1}^{n} h^0(\mathcal{O}(m-qd_i)) + h^0(\mathrm{Syz}(f_1^q, \ldots, f_n^q)(m)) \, .$$

Therefore we may express the Hilbert-Kunz function

$$\varphi(q) = \sum_{m=0}^{\infty} \lambda(R/I^{[q]})$$

as the alternating sum of this short exact sequence. Hence we apply the results of the previous section to this situation and deduce the following theorem.

Theorem 6.1. *Let K denote an algebraically closed field of positive characteristic p. Let R denote a normal two-dimensional standard-graded K-domain and let I denote a homogeneous R_+-primary ideal. Then the Hilbert-Kunz function of I has the form*

$$\varphi(q) = e_{HK}(I)q^2 + \gamma(q) \, ,$$

where $e_{HK}(I)$ is a rational number and $\gamma(q)$ is a bounded function.

Moreover, if K is the algebraic closure of a finite field, then $\gamma(q)$ is an eventually periodic function.

Proof. This follows at once from Theorem 5.1 applied to the short exact syzygy sequence. □

Remark 6.2. The rationality of the leading coefficient was proved in [2, Theorem 3.6]. That the Hilbert-Kunz function has the form $e_{HK}(I)q^2 + \beta q + \gamma(q)$ with real numbers $e_{HK}(I)$ and β and a bounded function $\gamma(q)$ was proved in [7]. In that paper Monsky also announces that $\beta = 0$. The periodicity was only known for the maximal ideal in the elliptic case ([3], [13], [5]), where the finiteness condition is not necessary. The periodicity in general for an arbitrary field is completely open.

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